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# A Probabilistic Model for Tracer Distribution in Multiphase Spatially Inhomogeneous Transport Systems

David Rappaport[1] and Joshua Dayan[2]

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A probabilistic model describing tracer transport in multiphase spatially inhomogeneous transport (plug-flow) systems is presented. The properties of the trajectories are completely described by a two-component Markov process with absorbing boundaries. The first component is continuous, the second discrete. Infinitesimal conditions are given. Probabilities associated with the process are derived.

**KEY WORDS:** Transport; plug flow; multiphase; tracer distribution; residence time distribution.

## 1. INTRODUCTION

The tracer technique has become a standard technique in the identification of flow systems and particle systems in the fields of chemical engineering,

[1] Faculty of Electrical Engineering, Technion-Israel Institute of Technology, Haifa, Israel.
[2] Faculty of Mechanical Engineering, Technion-Israel Institute of Technology, Haifa, Israel.

medicine, and physics. Testimony to this is the large amount of literature concerned with the subject (see, for example, Refs. 1–3).

Basically the technique consists in the injection of marked material (e.g., colored, radioactive) at one or several points (not simultaneously) into the system with arrival times being measured at other points. It is assumed that the injected material does not disturb or affect system behavior, i.e., the behavior of the marked material typifies the unmarked one.

Generally the point of view taken is one of the following:

(a) The tracer technique is used to experimentally determine the transfer function(s) [i.e., the arrival time distribution(s)] between the input(s) [injection point(s)] and the output(s) [measurement point(s)]. By using a finite number of such input–output pairs (transfer functions) the system is modeled by an $n$-dimensional lumped parameter system. Based on the form of the transfer function(s) (e.g., moments) a qualitative or quantitative judgement is made concerning the nature of the system. In particular the qualitative point of view is taken in the medical profession where the judgement is the state of health of the organ or body being examined. This idea is embodied in the frequently used Stewart–Hamilton technique.[4–7] More recent research embodying the above ideas can be found in Refs. 8–11. In particular Ref. 11 contains a lengthy bibliography of pertinent research.

(b) The system is assumed to be specified (i.e., by mathematical equations). Based on this assumption calculations are made to determine the distribution of tracer particles throughout the system as a function of time, including of course at the measurement points. The veracity of the assumed model is ascertained by comparison of the calculated arrival times with those experimentally obtained. This in particular is a point of view taken in physics and in chemical engineering.[12–14] As the particles move through space and time the system is described by a distributed parameter model.

In this work we shall adopt the second point of view. Our purposes are threefold:

(1) For the first time to provide a full, mathematically rigorous description of tracer distribution in the particular system of interest.

(2) To develop a framework which will lead to a system approach to identification. Toward the accomplishment of this goal we derive all relevant properties.

(3) To indicate how one can solve the partial difference equations appearing here. In particular in Section 4 we suggest a continuous time Monte Carlo simulation as opposed to more usual discrete time simulations and commonly employed numerical techniques.

We shall deal with flow systems in which there exist distinct flows (phases), each of which differs dynamically from the others. Further, there is an exchange of material between the different phases either by chemical or mechanical means. Some attempts have been made in the engineering literature to evaluate the arrival time distributions (also called residence time distributions, RTD) for such flow systems with more than one phase. This was done for the following cases: a stagnant phase and a flowing phase, a flowing phase and absorption, and two flowing phases (see Refs. 12–17). The solutions were obtained by writing a material balance for each phase and seeking a solution at the edges.

We note in passing that there are dual heat exchange systems in which the tracer is heat energy introduced into the system. In these systems temperature is measured at various points.

The class of systems considered in this work are commonly known as multiphase plug-flow (transport) systems in chemical engineering (physics). These systems are formally described by a finite set of coupled, first-order, hyperbolic partial differential equations. Each such equation describes a phase of the system and its connection with the other phases. We shall assume that the system is described on the subinterval $[a, b]$ of the real line and is time homogeneous. Each phase $i$ is characterized by a velocity profile $f(x, i)$ and exchange coefficients $\lambda_{ij}(x)$; each of the latter represents the time rate of exchange of material between the different phases $i$ and $j$ at the point $x \in [a, b]$. In Section 2 we present a complete description of the physical system along with a formal derivation of the partial differential equations by a material balance.

In all of the above-mentioned works a deterministic approach was used (i.e., material balance equations). We note, however, that the motion of a tracer particle is stochastic in nature and that the time–space distribution of many such noninteracting particles should be describable by the probability distribution associated with an appropriate random process.

In the present work a probabilistic model is suggested to describe tracer flow (in transport) in multiphase plug-flow systems. The phases are classified in the following manner: rest phases (stagnant), forward phases, and backward phases. Absorption or exit occurs at the forward end in forward phases and at the back end in backward phases (closed systems). In Section 3 the set of possible trajectories through the plug-flow systems for a tracer particle is described. Based on the physical description in Section 2, the tracer particle is assumed to follow a continuous path in a particular phase, its velocity being identical to that of the phase at the point at which the particle is located. The velocity profile for each phase is assumed to be a function of position (i.e., not necessarily constant). The particle at every instant decides whether or not to remain in its present phase and if not, to which other phase to jump. The decision to jump is based on the assumed infinitesimal properties of a postulated Markov transition probability. These infinitesimal conditions vary as a function of position.

In Section 4 properties of the process assumed in Section 3 are derived based on the postulated infinitesimal properties. A particular weak infinitesimal operator is derived. In Section 5 there is further discussion relating this work to other work.

A word on notation is appropriate. The symbol $\in$ is used to indicate belonging, and $\epsilon$ is used to denote an arbitrarily small, positive scalar quantity. The term exp signifies exponent. The symbol $E_t$ denotes mathematical expectation at time $t$. $I_B(x)$ represents the indicator function of the set $B$, i.e., $I_B(x) = 1$ if $x$ is in $B$, otherwise $I_B(x) = 0$. The symbolism $o(\Delta t)$, $\Delta t > 0$, represents a quantity of order of magnitude less than $\Delta t$, i.e., $\lim_{\Delta t \to 0} [o(\Delta t)/\Delta t] = 0$. All other notations are defined in the text.

## 2. DESCRIPTION OF THE PHYSICAL SYSTEM

We consider a conservative mass flow system defined on the subinterval of the real line $[a, b]$. We shall assume the existence of $m$ different interwoven parallel flows, called phases, each of which at time $t$ and at any cross section, i.e., $x \in [a, b]$, is characterized by a velocity $f(t, x, i)$ m/sec for $i = 1, ..., m$. That is, all particles belonging to a particular phase at cross section $x$ are moving with the same velocity. In chemical engineering this is known as "infinite radial diffusion" (i.e., a "white noise" type assumption which implies that particles are uniformly distributed within each phases and that present location within the phase cross section is independent of the past). We also suppose that at any $x \in [a, b]$ there exists a mechanism (e.g., chemical, mechanical) by which particles or material move from phase to phase. This mechanism will be represented by a positive exchange coefficient $\lambda_{ij}(t, x)$, whose units are $\sec^{-1}$ and which accounts for the fraction of the material in

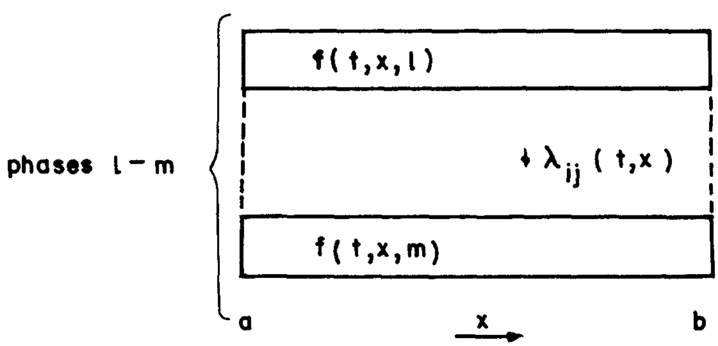


Fig. 1

phase $i$ moving to phase $j$, $i \neq j$, at time $t$ and cross section $x \in [a, b]$. Pictorially we represent this in Fig. 1.

The phases will be restricted to one of the following three categories: forward, backward, and rest. Forward and backward phases shall be called dynamic phases. Rest phases are those for which $f(t, x, i) = 0$ for all $t$ and $x \in [a, b]$. Further, it is assumed that there exists an $f_{\min} > 0$ such that if $i$ is a forward (backward) phase $f(t, x, i) > f_{\min}$ ($< -f_{\min}$) for all $t$ and $x \in [a, b)$ ($(a, b]$). For the case of forward (backward) phases new material enters the system from outside at $x = a$ ($x = b$) and exits at $x = b$ ($x = a$). For the case of rest phases no new material enters from without. At exits the material is collected in some fashion.

Continuing, we shall formally derive the transport equations which describe the aforementioned system.

Let $F(t, x, i)$ be the total amount of material in phase $i$ at time $t$ in the interval $[a, x]$. Let $k_i(t)$ be the rate at which new material is entering dynamic phase $i$. Let $\Delta t > 0$ be an arbitrary small interval of time. We use $\Delta t$ to define the following partitions of the various phases: For a forward (backward) phase $i$ we form the spatial partition $\{a, x_1^i, x_2^i, \ldots, b\}$ by the recursion $x_{l+1}^i = x_l^i + f(t, x_l^i, i)\,\Delta t$ ($x_{l+1}^i = x_l - f(t, x_l^i, i)\,\Delta t$), where $x_0^i = a$: For a rest phase the partition is generated by $x_{l+1}^i = x_l^i + \alpha\,\Delta t$, where $\alpha > 0$ is arbitrary. With the aid of Fig. 2, which illustrates the case of a forward phase, we heuristically formulate the equations of mass balance.

$$\begin{aligned} F(t + \Delta t, x, i) = {} & F(t, x - f(t, x, i)\,\Delta t, i) \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{l=0}^{[x^j] \leq x} \lambda_{ji}(t, x_l^j)\,\Delta t\,[F(t, x_{l+1}^j, j) - F(t, x_l^j, j)] \\ & - \sum_{l=0}^{[x^i] \leq x} \lambda_{ii}(t, x_l^i)\,\Delta t\,[F(t, x_{l+1}^i, i) - F(t, x_l^i, i)] \\ & + \int_t^{t+\Delta t} k_i(\tau)\,d\tau + o(\Delta t) \end{aligned}$$

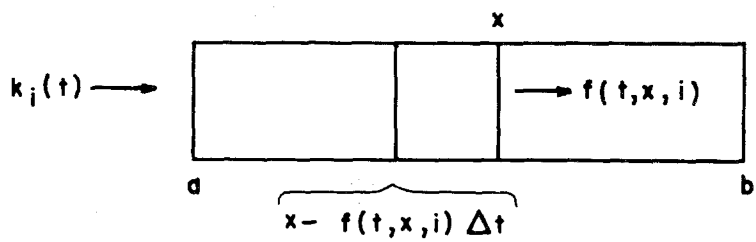


Fig. 2

where $[\ ] \leqslant x$ denote the largest element in the partition less than $x$, and $\lambda_{ii} = \sum_{j=1, j\neq i}^{m} \lambda_{ij}$. Note that the second term on the right represents the total material passing from phases other than $i$ to $i$ during time $\Delta t$, while the third term on the right denotes the total material leaving $i$ for other phases during $\Delta t$.

Letting $\Delta t \to 0$, differentiating both sides with respect to $x$, and using the notation $C(t, x, i) = \partial F(t, x, i)/\partial x$ to denote the concentration per unit length of material in $i$ at time $t$ and position $x$, one formally obtains the following set of hyperbolic partial differential equations (utilizing the obvious shortened notations):

$$\frac{\partial C_i}{\partial t} = -\frac{\partial}{\partial x}(f_i C_i) + \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{m} \lambda_{ji} C_j \right\} - \lambda_{ii} C_i \tag{1}$$

for $i = 1, \ldots, m$.

In the general case both $f_i$ and $\lambda_{ij}$ are assumed to be functions of the different local concentrations and their derivatives and thus Eq. (1) is nonlinear.

Here we shall assume that the velocities and exchange coefficients are not explicitly time dependent and that a time-independent (steady-state) solution exists for Eq. (1), subject of course to the appropriate time-invariant boundary conditions (i.e., $k_i$ constant for each $i$). We shall further suppose that the steady-state solution is asymptotically stable.

The introduction of marked tracer material into such a system is presumed not to disturb the steady-state solution and a mass balance such as provided above, but this time only for the tracer, results again in $m$ equations of the form of (1). However, in this case since the velocities and exchange coefficients are supposedly unperturbed from their steady-state values, we can express them as explicit functions of position $x$ and phase coordinates. This is entirely analogous to the point of view taken in shock theory in which the shock dynamics are taken to be the first-order (i.e., linear) variation about a steady-state solution.

The purpose of the injection of tracer into the system is generally to determine experimentally behavior or parameters of the systems which cannot be directly measured.

Let $C(t, x, y)$ denote a solution to (1) for the tracer concentration for $x \in [a, b]$, $y \in \{1, \ldots, m\}$. Let $g(x, y)$ denote a bounded function of the two variables. Proceeding formally, multiply both sides of Eq. (1) for phase $i$ by $g(x, i)$; integrate with respect to $x$, sum over the $i$'s; interchange the order of differentiation and integration freely; use integration by parts and assume that the product $gfC$ is zero at the system boundaries. This formal manipulation results in

$$\frac{d}{dt}\int_a^b \sum_{i=1}^m g(x, i)C(t, x, i)\,dx$$

$$= \int_a^b \sum_{i=1}^m \left[ f(x, i)\frac{\partial g(x, i)}{\partial x} + \sum_{\substack{j=1 \\ j\neq i}}^m g(x, j)\lambda_{ij}(x) - g(x, i)\lambda_{ii}(x) \right] C(t, x, i)\,dx \tag{2}$$

subject of course to appropriate boundary conditions.

In the next section we postulate the existence of a two-component Markov process with discrete second component. In Section 4 we identify the terms in Eq. (2) with a weak infinitesimal operator (generator) of this process.

## 3. THE MARKOV TRACER MODEL FOR AN $m$-PHASE PLUG-FLOW SYSTEM

**A.** We denote the space of sample functions of the process by $\Omega$. The letter $z$ will denote the vector $[x, y]$, where $x$ takes values in the closed interval of the real line $[a, b]$, $a < b$, and $y$ takes values in the set of $m$ points $Y = \{1, \ldots, m\}$. Each point in $Y$ shall be called a phase. $H$ will symbolize the state space of the process which is the product space $[a, b] \times Y$.

$\Omega$ is the set of all the functions $z(t, \omega)$ taking values in $H$, where $t$ belongs to the semiinfinite interval $[0, \infty)$ and where $y(t, \omega)$ is a function in the space of all right continuous step functions on $[0, \infty)$ taking values in $Y$ with a finite number of isolated jumps (i.e., changes in value) on every bounded subinterval $[t_1, t_2]$, $0 \leqslant t_1 < t_2 < \infty$. The index $t$ shall be interpreted as time. On $[0, \infty)$, $x(t, \omega)$ satisfies the differential equation

$$\dot{x}(t, \omega) = f(x(t, \omega), y(t, \omega)) \tag{3}$$

with the initial condition $x(0, \omega) = x_0$ in $[a, b]$. The functions $f(x, j)$ for $j \in Y$ are uniformly bounded in $x$ on $[a, b]$. We shall arbitrarily assume that the phase $j$ for $1 \leqslant j \leqslant l_1$ is a rest phase, for $l_1 + 1 \leqslant j \leqslant l_2$ a forward phase, and for $l_2 + 1 \leqslant j \leqslant m$ a backward phase (see Section 2 for the velocity properties defining the different phases). In addition we shall assume that for a forward (backward) phase $i$, $f(x, i)$ is at least Lipschitz continuous in $x$ on $[a, b)$ $((a, b])$ and that $f(b, i) = 0$ $(f(a, i) = 0)$. Let $f_{\max} = \max_{z\in H}|f(x, y)|$.

By the solution to Eq. (3) on any time interval $[0, T]$ we shall mean the solution obtained by joining (by continuation) the solutions on the intervals between discontinuities in the derivative of $x(t, \omega)$, using the limiting state from the left (i.e., before the discontinuity) as the initial state for the interval commencing with the time of the discontinuity in the derivative. Clearly the solution to (3) is continuously differentiable except at a finite number of points on $[0, T]$, at which the derivative is continuous from the right. It follows from the above description that $x(t, \omega)$ is confined to the interval $[a, b]$ for all $t \geqslant 0$.

For each $\omega$ we shall interpret each $x(t, \omega)$ for $t \geqslant 0$ as one of the possible trajectories of a tracer particle with the jump times and the phases visited given by the function $y(t, \omega)$.

**B.** We take as the $\sigma$-field of events on $\Omega$, denoted $\beta$, the one generated by all finite base cylinder sets of the type

$$\{\omega: \quad z(t_1, \omega) \in A_1, \ldots, z(t_n, \omega) \in A_n\}$$

for every finite subset of times $t_i$ belonging to $[0, \infty)$ and for arbitrary events $A_i$ belonging to the $\sigma$-field in $H$, denoted $\mathscr{H}$, generated by product sets of the form $[c, d] \times \{i\}$, where $a \leqslant c \leqslant d \leqslant b$, and $i$ belongs to $Y$.

One can verify that sets of the form

$$\{\omega: \quad y(t, \omega) \text{ jumps exactly } n \text{ times on } [0, T]\}$$

are events in $\beta$ for each $n$, $n = 0, 1, \ldots$, and $T \geqslant 0$.

We shall assume the following properties for the exchange coefficients $\lambda_{ij}$. For the case of a rest phase $i$ we require $\lambda_{ij}(x)$ to be continuous for all $x \in [a, b]$ and for each $j \neq i$. For $i$ a forward (backward) phase for each $j \neq i$ we take $\lambda_{ij}(x)$ to be piecewise continuous bounded on $[a, b)$ ($(a, b]$) with at most a finite number of discontinuities. At a point of discontinuity we require $\lambda_{ij}(x)$ to be continuous from the right (left). At $x = b$ ($x = a$), $\lambda_{ij}(b) = 0$ ($\lambda_{ij}(a) = 0$).

**C.** We postulate the existence of a time stationary Markov transition function on the state space $H$ and $\sigma$-field $\beta$ which generates a strong Markov process on $(\Omega, \beta)$ possessing the infinitesimal properties stated below. For $A \in \mathscr{H}$ we shall denote this transition function by $P(\tau, A|z)$ where $\tau \geqslant 0$ and $z \in \mathscr{H}$. (See Dynkin,[22] Chapter II.) By generation of the Markov process by the Markov transition probability we mean that for $t \geqslant s$ the probability of being at time $t$ in set $A$, conditioned on being at point $z$ at time $s$, is given by

$$P(t, A|s, z) = P(t - s, A|z)$$

Let $i$ be a forward (backward) phase and the set of discontinuities on $[a, b)$ (on $(a, b]$) of $\lambda_{ij}(x)$ for $j \neq i$ be $\rho_k^{ij}$, $k = 1, \ldots, n_{ij}$. We then assume that for each $\Delta t > 0$ and $j \neq i$

$$|P(\Delta t, [a, b] \times \{j\}|x, i) - \lambda_{ij}(x)\Delta t| \leqslant o(\Delta t)$$

uniformly for all $x$ outside of the subintervals $[\rho_k^{ij} - f_{\max}\,\Delta t, \rho_k^{ij}]$ ($[\rho_k^{ij}, \rho_k^{ij} + f_{\max}\,\Delta t]$), for $k = 1, ..., n_{ij}$. Clearly this implies that the probability of jumping to phase $j$ from phase $i$, position $x$, on a time interval of duration $\Delta t$ is of the order of $\lambda_{ij}(x)\,\Delta t$.

For the case of $i$ a rest phase we assume that the last relation is true uniformly for all $x \in [a, b]$. Second, we assume that there exists a positive constant $L$ such that

$$P(\Delta t, [a, b] \times \{j\}|x, i) \leqslant L\,\Delta t$$

uniformly for all $z \in H$. Third we require that the probability

$$P(\text{two or more jumps in time } \Delta t|z) \leqslant o(\Delta t)$$

uniformly for all $z \in H$.

We can interpret the function $\lambda_{ij}$ as local mean time to jump from $y = i$ to $y = j$ given the state $z = [x, i]$.

We wish to *strongly emphasize* the fact of the assumption allowing for discontinuities in the exchange coefficients as described above.

We now proceed in Section 4 to use the infinitesimal properties to further characterize the process.

## 4. PROCESS PROPERTIES

Assuming the existence of the postulated transition probability function in Section 3, we proceed to develop formulas one of which is the probability that there will be no jump on a specified interval given the initial state at the beginning of the interval, the other being the probability of jumping from phase $i$ to phase $j$ on a specified interval given the initial state $z$. Second, we derive a weak infinitesimal operator of the process and specify its domain. We conclude this section with a discussion of additional properties possessed by the process.

**Theorem 1.** For each $t \in [0, \infty)$, $\alpha > 0$,

$$P(y_{t+\tau} = y_t \text{ each } \tau \in [0, \alpha]|t, z) = \exp\left[-\int_t^{t+\alpha} \lambda_{yy}(x_s)\,ds\right]$$

where $z = [x, y]$ and $x_s$ is the solution to the differential equation on $[t, t + \alpha)$

$$\dot{x}_s = f(x_s, y), \qquad x_t = x$$

The proof is given in Appendix A.

We remark that if we take $\gamma$ to be the time to the first jump after time $t$, then

$$P(\gamma \geqslant \alpha|t, z) = P(\text{no jump on } [t, t + \alpha|t, z) = \exp\left[-\int_t^{t+\alpha} \lambda_{yy}(x_s)\,ds\right]$$

**Corollary to Theorem 1.** For $a \leqslant x_1 \leqslant x_2 \leqslant b$ $(a \leqslant x_2 \leqslant x_1 \leqslant b)$ and $y$ a forward (backward) phase

$$P(\text{reaching } x_2 \text{ without jumping } | x_1, y) = \exp\left[-\int_{x_1}^{x_2} \frac{\lambda_{yy}(v)}{f(v, y)}\, dv\right]$$

*Proof.* The proof follows from Theorem 1, time stationarity, and the properties assumed for $f(x, y)$.

Theorem 1 can also be derived by the method described by Skorokhod,[18] who treats processes related to this work.

Theorem 1 and its corollary provide the basis for an exceedingly simple method for Monte Carlo simulations of our transport process, in particular for the case of an inhomogeneous system. In order to find the location of the next jump point, one simply transforms the output of a random number generator into the jump point by means of the corollary and a standard technique. By the strong Markov property one generates a series of jump points. At each jump point $x$ a jump to phase $j$ from $i$ is made according to the probability $\lambda_{ij}(x)/\lambda_{ii}(x)$ for $j \neq i$ (see Theorem 2). We contrast this to the more common multidecision Monte Carlo techniques which entail stopping the process on a grid of specified times at each of which a decision is made whether or not the tracer particle remains in the present phase or moves to another.

Application of the technique suggested above to the solution of first-order hyperbolic partial differential equations and comparison of this technique to a standard method, the method of characteristics, is being made. It appears that the more inhomogeneous the system, the greater the unreliability of the method of characteristics.

Using the infinitesimal conditions and the last result, one can derive the conditional probability of jumping from $i$ to $j$ on a finite interval. This result is stated below. The explicit form in the theorem is exploited in the proof of results related to tracer identifiability.[19]

**Theorem 2.**

$$P(y(t + \alpha) = j \neq i, \text{ one jump on } [t, t + \alpha] | t, x, i)$$

$$= \int_t^{t+\alpha} \exp\left[-\int_t^s \lambda_{ii}(x_\gamma)\, d\gamma\right] \lambda_{ij}(x_s) \exp\left[-\int_s^{t+\alpha} \lambda_{jj}(x_\beta{}^*)\, d\beta\right] ds$$

where $x_\gamma$ is the solution to

$$\dot{x}_\gamma = f(x_\gamma, i), \qquad \gamma \in [t, s)$$
$$x_t = x$$

and $x_\beta{}^*$ is the solution to

$$\begin{aligned} \dot{x}_\beta{}^* &= f(x_\beta{}^*, j), \qquad \beta \in [s, t + \alpha] \\ x_s{}^* &= x_s \end{aligned}$$

The proof is given in Appendix B.

As a check, one can work backward to derive the infinitesimal conditions from Theorems 1 and 2.

We shall let the symbol $g(x)$ denote the vector

$$g(x) = \begin{bmatrix} g(x, i) \\ \vdots \\ g(x, m) \end{bmatrix}$$

where $g(x, i)$, $i = 1, ..., m$, are functions belonging to the Banach space $B(H)$ of bounded, measurable functions on $H$ with the norm $\sup_H |g(x, y)|$.

Without further comment we note that in the sequel we shall use both the scalar and vector notations, the choice being simply a question of convenience in the particular instance. For $\Delta t > 0$ we shall now investigate the operation

$$\lim_{\Delta t \to 0} \frac{E_{t+\Delta t}(g|t, z) - g(z)}{\Delta t}$$

where

$$E_{t+\Delta t}(g|t, z) = \sum_{j=1}^{m} \int g(x, j)\, d_\alpha P(t + \Delta t, x \leqslant \alpha, y = j|t, z)$$

where $z = [x, y]$, and $g(x, y) \in B(H)$. If the pointwise limit exists, we shall denote it by $Ag(z)$, $A$ being the weak infinitesimal operator of the process on $B$ (see Dynkin,[22] Chapters I and II). $D_A$, the domain of $A$ in $B$, will denote the subset of functions in $B$ for which the pointwise limit exists and is bounded for all $z \in H$. Continuing,

$$\begin{aligned} &\frac{E_{t+\Delta t}(g|t, z) - g(z)}{\Delta t} \\ &\quad = \frac{1}{\Delta t}\left[g\left(x + \int_t^{t+\Delta t} f(x_\gamma, y)\, d\gamma, y\right)\right] P(\text{no jump on } [t, t + \Delta t]|t, z) \\ &\qquad + \frac{1}{\Delta t} \sum_{\substack{j=1 \\ j \neq y}}^{m} E(I_{C_{yj}} g(x_{t+\Delta t}, j)|t, z) - \frac{1}{\Delta t} g(z) + \frac{o(\Delta t)}{\Delta t} \end{aligned} \tag{4}$$

where

$$C_{yj} = \{\omega\colon \quad x_t = x, y_t = y, \text{one jump to } j \neq y \text{ on } [t, t + \Delta t]\}$$

and $x_\gamma$ for $\gamma \in [t, t + \Delta t]$ is the solution to $\dot{x}_\gamma = f(x_\gamma, y)$, $x_t = x$.

Noting that with probability one $x_t$ for $t \geqslant 0$ is restricted to $[a, b]$, we let the term $o(\Delta t)$ uniformly in $z$ take care of the case of two or more jumps on $[t, t + \Delta t]$.

We now consider the various types of phases.

(a) *Let $y$ be a rest phase.* Let $\Delta t > 0$ be arbitrary. By the path properties and the infinitesimal conditions the right side of Eq. (4) becomes

$$\frac{1}{\Delta t} g(x, y)[1 - \lambda_{yy}(x)\,\Delta t + o(\Delta t)] - \frac{1}{\Delta t} g(x, y)$$

$$+ \frac{1}{\Delta t} \sum_{\substack{j=1 \\ j \neq y}}^{m} E(I_{C_{yj}} g(x_{t+\Delta t}, j) | t, z) + \frac{o(\Delta t)}{\Delta t}$$

Also for $j \neq y$

$$\begin{aligned} E_{t+\Delta t}(I_{C_{yj}} g | t, z) &\leqslant \Delta t\, \lambda_{yj}(x) \sup_{\alpha \in \phi_j} g(x + f_{\max}\alpha, j) + o(\Delta t) \\ &\geqslant \Delta t\, \lambda_{yj}(x) \inf_{\alpha \in \phi_j} g(x + f_{\max}\alpha, j) + o(\Delta t) \end{aligned} \tag{5}$$

where the $o(\Delta t)$ terms are uniformly bounded in $x$ and the set

$$\phi_j = \left\{\alpha\colon\ 0 \leqslant \alpha \leqslant \max\left(\Delta t, \frac{b - x}{f_{\max}}\right)\right\} \qquad \text{for } j \text{ forward}$$

or

$$\phi_j = \left\{\alpha\colon\ \min\left(-\Delta t, \frac{x - a}{f_{\max}}\right) \leqslant \alpha \leqslant 0\right\} \qquad \text{for } j \text{ backward}$$

Thus by taking $g(x, i)$ to be right (left) continuous for $j$ forward,

$$\lim_{\Delta t \to 0} \frac{E_{t+\Delta t}(g | t, z) - g(z)}{\Delta t} = -\lambda_{yy}(x) g(x, y) + \sum_{\substack{j=1 \\ j \neq y}}^{m} \lambda_{yj}(x) g(x, j) \tag{6}$$

independently of $t$, uniformly in $x$.

(b) *Let $y$ be a forward phase.* Choose $\Delta t > 0$ such that $x < \rho_k^{yj} - f_{\max}\,\Delta t$ for each $k$, $k = 1, \ldots, n_{yj}$, and for each $j \neq y$ (i.e., $\Delta t$ sufficiently small such that on $[t, t + \Delta t]$ the trajectory does not encounter a point of discontinuity of $\lambda_{ij}$).

Rewriting Eq. (4) and using the infinitesimal conditions

$$\begin{aligned} &E_{t+\Delta t}(g | t, z) - g(z) \\ &\quad = \frac{g(x + \int_t^{t+\Delta t} f(x_\gamma, y)\, d\gamma, y) - g(x, y)}{\Delta t} \\ &\qquad - \lambda_{yy}(x) g\left[x + \int_t^{t+\Delta t} f(x_\gamma, y)\, d\gamma, y\right] \\ &\qquad + \frac{1}{\Delta t} \sum_{\substack{j=1 \\ j \neq y}}^{m} E_{t+\Delta t} I_{C_{yj}} g | t, z) + \frac{o(\Delta t)}{\Delta t} \end{aligned} \tag{7}$$

Assume that $g(x, y)$ has a piecewise continuous derivative with respect to $x$ with at most a finite member of discontinuities, and right continuous at the points of discontinuity. Further restrict $\Delta t$ such that the derivative is continuous on $[x, x + f_{\max} \Delta t)$. For each such $\Delta t$ apply the mean value theorem to the first term on the right side of (7) twice, once to $g$ and once to $\int_t^{t+\Delta t} f(x_\gamma, y)\, d\gamma$. We thus obtain

$$\frac{g(x + \int_t^{t+\Delta t} f(x_\gamma, y)\, d\gamma, y) - g(x, y)}{\Delta t}$$
$$= g'\left(x + \Theta_1 \int_t^{t+\Delta t} f(x_\gamma, y)\, d\gamma, y\right) f(x_{t+\Theta_2 \Delta t}, y) \tag{8}$$

where $0 < \Theta_i < 1$ for $i = 1, 2$ and $g'$ denotes the partial derivative with respect to $x$.

From the path properties and the infinitesimal properties for the values of $\Delta t > 0$ given above we conclude that the third term on the right side of (7) is bounded from above and below as in (5). Therefore by (8), (5), and the continuity of $g, g', f$ and of the trajectory $x_t$ at $x$

$$\lim_{\Delta t \to 0} \frac{E_{t+\Delta t}(g|t, z) - g(z)}{\Delta t}$$
$$= f(x, y) \frac{\partial^+ g(x, y)}{\partial x} - \lambda_{yy}(x) g(x, y) + \sum_{\substack{j=1 \\ j \neq y}}^{m} \lambda_{yj}(x) g(x, j) \tag{9}$$

where $+$ indicates derivative from the right.

For $x = b$ it follows immediately from the path properties that Eq. (4) equals zero for each $\Delta t > 0$.

(c) *Let $y$ be a backward phase.* Choose $\Delta t > 0$ such that $x > \rho_k^{yj} + f_{\max} \Delta t$ for each $k$, $k = 1, ..., n_{yj}$, and for each $j \neq y$. Assume that $g(\cdot, y)$ has a finite, piecewise continuous derivative with at most a finite number of discontinuities and left continuous at point of discontinuity. Proceeding as for case (b), we obtain an equation similar to (9) with the only exception being that the right derivative with respect to $x$ is replaced by the left derivative.

For $x = a$ one can verify that Eq. (4) equals zero for each $\Delta t > 0$.

Imposing the notation convention that $f(x, y) = 0$ implies that $f(x, y)\, \partial g(x, y)/\partial x = 0$, we sum up the above in the following theorem.

**Theorem 3.** The weak infinitesimal operator $A$ in $B$ of the Markov process is given by

$$Ag(z) = f(x, y) \frac{\partial g(x, y)}{\partial x} - g(x, y) \lambda_{yy}(x) + \sum_{\substack{j=1 \\ j \neq y}}^{m} g(x, j) \lambda_{yj}(x) \tag{10}$$

The domain of $D_A$ is given by the set of functions $g(x, y)$ on $H$ which have finite, piecewise continuous derivatives with respect to $x$ with at most a finite number of discontinuities, right (left) continuous at the points of discontinuity for $y$ a forward (backward) phase.

*Remark.* Obviously the state $(b, y)$ $((a, y))$ where $y$ is a forward (backward) phase is an absorbing state (see Dynkin,[22] p. 137, lemma 5.3). This is expressing mathematically the intuitively obvious fact that upon reaching the right (left) boundary in a forward (backward) phase the path remains at that boundary for all times thereafter.

We notice that as a function of $y$, Eq. (10) represents $m$ coupled equations. Therefore, we shall conveniently represent the operator $A$ in the vector form given below.

Let

$$g(x) = \begin{bmatrix} g(x,1) \\ \vdots \\ g(x,m) \end{bmatrix}, \qquad F(x) = \begin{bmatrix} f(x,1) & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & f(x,m) \end{bmatrix}$$

and

$$\Gamma(x) = \begin{bmatrix} -\lambda_{11}(x)\lambda_{12}(x) \cdots & \lambda_{1m}(x) \\ \vdots & \vdots \\ \lambda_{m1}(x)\lambda_{m2}(x) \cdots & -\lambda_{mm}(x) \end{bmatrix}$$

Therefore

$$Ag(x) = F(x)\frac{dg(x)}{dx} + \Gamma(x)g(x) \tag{11}$$

where we take $Ag(x)$ to mean

$$\begin{bmatrix} Ag(x,1) \\ \vdots \\ Ag(x,m) \end{bmatrix}$$

It has been assumed throughout that in forward (backward) phases the velocities must be greater than some strictly positive (negative) value, except at the right (left) boundary as described in Section 2. From a strictly mathematical point of view the case for which the velocities approach zero can be treated given proper assumptions (i.e., $\int_a^b [\lambda_{yy}(v)/f(v,j)]\,dv$ finite for each $j$) by methods such as in Chapters 4 and 5 of Ref. 23.

By Dynkin,[22] paragraph 1.15, p. 40, for $g \in D_A$ the expectation $E_t(g|z_0)$ is continuous from the right for every $z_0 \in H$, and in addition the time derivative from the right exists, is continuous for every $z_0 \in H$, and satisfies

$$\partial^+ E_t(g|z_0)/\partial t = AE_t(g|z_0) \tag{12a}$$

$$= E_t(Ag|z_0) \tag{12b}$$

where $\lim_{t\to 0} E_t(g|z_0) = g(z_0)$.

In (12a), $A$ in vector form is used operating on the variable $z_0$, while in (12b), $A$ in scalar form (10) is used for each $z_0$. Suppose that we choose $g(x, y) \in D_A$ such that for $j$ a forward phase

$$g(x, y) = \begin{cases} 1, & \alpha \leqslant x < \beta, \quad y = j \\ 0, & \text{otherwise} \end{cases}$$

where $a < \alpha < \beta \leqslant b$. Then by (12a)

$$\frac{\partial^+ P(t, \alpha \leqslant x < \beta, j | z_0)}{\partial t} = F(x_0) \frac{\partial P(t, \alpha \leqslant x < \beta, j | z_0)}{\partial x_0} + \Gamma(x_0) P(t, \alpha \leqslant x < \beta, j | z_0) \tag{13a}$$

Similarly for $j$ a backward phase we obtain

$$\frac{\partial^+ P(t, \alpha < x \leqslant \beta, j | z_0)}{\partial t} = F(x_0) \frac{\partial P(t, \alpha < x \leqslant \beta, j | z_0)}{\partial x_0} + \Gamma(x_0) P(t, \alpha < x \leqslant \beta, j | z_0) \tag{13b}$$

Similar equations hold for rest phases. By (12b) for $j$ a forward phase

$$\frac{\partial^+ P(t, \alpha \leqslant x < \beta, j | z_0)}{\partial t} = -\int_\alpha^\beta \lambda_{jj}(\gamma)\, d_\gamma P(t, x < \gamma, j | z_0) + \sum_{\substack{y=1 \\ y \neq j}}^{m} \int_\alpha^\beta \lambda_{yj}(\gamma)\, d_\gamma P(t, x < \gamma, y | z_0) \tag{14a}$$

For constant exchange coefficients on $(a, b)$ this last equation becomes

$$\frac{\partial^+ P(t, \alpha \leqslant x < \beta, j | z_0)}{\partial t} = -\lambda_{jj} P(t, \alpha \leqslant x < \beta, j | z_0) + \sum_{\substack{y=1 \\ y \neq j}}^{m} \lambda_{yj} P(t, \alpha \leqslant x < \beta, y | z_0) \tag{14b}$$

This equation states that the net amount of material entering during a short time interval the region $[\alpha, \beta)$ in phase $j$ via phase $j$ is negligible compared to that crossing into phase $j$ from other phases on the subinterval $[\alpha, \beta)$.

## 5. DISCUSSION

By formal substitution of the spatial derivative of probability distribution function $F(t, \alpha, j | z_0) = P(t, x < \alpha, y = j | z_0)$ into Eqs. (12a) and (12b) one obtains (2). Essentially this is what was done in Ref. 20 to obtain forward and backward equations of Kolmogorov related to this process.

An example of a degenerate system (i.e., one phase without absorption) is discussed in Ref. 22, paragraph 2.15, p. 62. In Ref. 18 processes of the type considered here are discussed. Theorem 1 is derived by another method. However, the problem of absorption, the properties of the exchange coefficients, and the description of $D_A$ are not considered.

For the case of all phases being rest phases the model reduces to the case of jump Markov processes such as discussed by Gikhman and Skorokhod[21] as well as in other texts. For the homogeneous case of fixed velocities and fixed exchange coefficients on $(a, b)$ the system is equivalent to a process whose paths are integrals of the sample functions of a renewal process in which the time scale varies from phase to phase.

In future work we shall exploit Theorems 1–3 and Eqs. (12a) and (12b) in order to specify conditions which ensure identifiability of the system from input–output data. We plan to investigate properties of the residence time distributions (i.e., output data) in order to develop an explicit method for identification of the spatially varying system parameters. Further, we shall examine the case of systems with recycling (i.e., reflecting boundaries).

Development of novel, more efficient schemes for the computation of the solution of the linear hyperbolic partial differential equation describing multiphase plug-flow is currently under way.

## APPENDIX A. PROOF OF THEOREM 1

Assume for $\tau \in [t, t + \alpha]$ that $\lambda_{yj}(x_\tau)$ is continuous for all $j \neq y$ with the possible exception being at $t + \alpha$ ($x_{t+\alpha}$ is defined in the statement of the theorem). Suppose that $y$ is a forward phase.

Let $\gamma$ be the unique solution (time) which satisfies the equation $\dot{r}_s = f(r_s, y)$, $s \in [0, \gamma]$, for given initial condition $r_0$ and final condition $r_f$, i.e., the time it takes to go from $r_0$ to $r_f$ in phase $y$. The unidirectional nature of the velocity $f$ ensures a unique $\gamma$ for which $r_\gamma = r_f$, provided the initial and final conditions are consistent with the sign of $f$. In particular let $\gamma$ be the solution for $r_0 = x_{t+\alpha} - f_{\max}\,\Delta t$ and $r_f = x_{t+\alpha}$. It therefore follows from the infinitesimal conditions that

$$P(y(\tau, \omega) = y \text{ all } \tau \in [0, \gamma] | x_{t+\alpha} - f_{\max}\,\Delta t, y) \geqslant 1 - L\gamma - o(\gamma)$$

Since $\Delta t \to 0$ implies that $\gamma \to 0$ for $\epsilon > 0$, choose $\Delta t$ in the sequel such that the above probability $\geqslant (1 - \epsilon)$. Let $\rho \in [t, t + \alpha]$ be arbitrary. Let

$$A = \{\omega\colon\ \ y(\tau, \omega) = y \text{ for } \tau \in [t, \rho],\quad x(t, \omega) = x\}$$

$$B = \{\omega\colon\ \ y(\tau, \omega) = y \text{ for } \tau \in [\rho, t + \alpha],\quad x(\beta, \omega) = x + \int_t^\rho f(x_s, y)\, ds\}$$

$$T = A \cap B = \{\omega\colon\ \ y(\tau, \omega) = y,\quad \tau \in [t, t + \alpha],\quad x(t, \omega) = x\}$$

Note that by the right continuity of the sample paths it follows that $A$, $B$, and $T$ are generated by cylinder sets on $[t, t + \alpha]$ and hence measurable events. Thus by the Markov property

$$P(T|t, x, y) = P\left(B|\rho, x + \int_t^\rho f(x_s, y)\, ds, y\right) P(A|t, x, y)$$

Let $\{t_l^n\}$, $l = 0, 1, ..., 2^n$, be the $n$th binary partition of $[t, t + \alpha - \gamma]$ where $t_0^n = t$, $t_{2^n}^n = t + \alpha - \gamma$.

Inductively we surmise that

$$\begin{aligned} P(T|t, x, y) &= \left[\prod_{l=0}^{2^n-1} P\left(y(\tau, \omega) = y \text{ all } \tau \in [t_l^n, t_{l+1}^n] | t_l^n, x + \int_t^{t_l^n} f(x_s, y)\, ds, y\right)\right] \\ &\quad \times [1 - O(\epsilon)] \end{aligned}$$

where $O(\epsilon)$ goes to zero as $\epsilon$ goes to zero.

Taking natural logarithms and using the infinitesimal properties and letting $\Delta t^n = (\alpha - \rho)/2^n$, we obtain

$$\ln P(T|t, x, y) = \sum_{l=0}^{2^n-1} \ln[1 - \lambda_{yy}(x(t_l^n))\, \Delta t^n + o(\Delta t^n)] + \ln[1 + O(\epsilon)]$$

where $x(t_l^n)$ is the solution to $\dot{x}_s = f(x_s, y)$, $x_t = x$, $s \in [t, t_l^n]$, and $o(\Delta t^n)$ uniformly bounded on $[a, b] \times Y$. Recalling that for $-1 < p < 1$, $\ln(1 + p) = p + O(p^2)$, by the continuity of $\lambda_{yy}$ on $[t, t + \alpha - \rho]$ we obtain

$$\ln P(T|t, x, y) = -\sum_{l=0}^{2^n-1} \lambda_{yy}(x(t_l^n))\, \Delta t^n + 2^n o(\Delta t^n) + O(\epsilon)$$

Since $\epsilon$ is arbitrary and $\lim_{n\to\infty} 2^n o(\Delta t^n) = 0$, passing to the limit as $n \to \infty$, we obtain the result for the case of $y$ a forward phase and a possible discontinuity of $\lambda_{yy}$ at $x_{t+\alpha}$ (recall that $\Delta t$ is arbitrary).

For the case of multiple finite discontinuities in $\lambda_{yy}$, we let $x_{\tau_i}$, $i = 1, \ldots n_{yy}$ be the points of discontinuity and $\tau_i$ the times on the trajectory $x_\tau = x + \int_t^\tau f(x_s, y)\, ds$, $\tau \geqslant t$, at which the discontinuities are encountered. Thus as shown above by the Markov property

$$P(T|t, x, y) = \prod_{i=0}^{n_{yy}} P(y(\tau, \omega) = y, \text{ all } \tau \in [\tau_i, \tau_{i+1}] | \tau_i, x_{\tau_i}, y)$$

where $\tau_0 = t$, $t_{n_{yy}+1} = t + \alpha$. Then the result is obtained by using the result above for the case of a discontinuity at the endpoint.

For $y$ a backward phase the same proof holds with the obvious changes. The case of $y$ a rest phase is actually a subcase of the forward case.

## APPENDIX B. PROOF OF THEOREM 2

All the events defined below are measurable (see explanation in the proof of Theorem 1, Appendix A).

Let $\rho_l$ be the finite, ordered set of discontinuities in the transfer coefficients $\lambda_{ij}(x)$ for $j = 1, \ldots, m$. Suppose that $i$ is a forward phase. Let $\tau_l$ be the unique times on the trajectory $x_s$, $s \in [t, t + \alpha]$ for which $x_{\tau_l} = \rho_l$ (see theorem statement). For $j \neq i$ let

$$\begin{aligned} C = \{\omega\colon \quad & x(t, \omega) = x,\ y(t, \omega) = i, \\ & y(t + \alpha, \omega) = j \neq i, \text{ one jump on } (t, t + \alpha]\} \end{aligned}$$

Let

$$\begin{aligned} C_l = \{\omega\colon \quad & x(t, \omega) = x,\ y(\tau, \omega) = i,\ \tau \in [t, \tau_l]; \\ & y(\tau, \omega) = j,\ \tau \in [\tau_{l+1}, t + \alpha]; \text{ one jump on } (\tau_l, \tau_{l+1})\} \end{aligned}$$

Clearly the $C_l$ are disjoint and $C = \bigcup C_l$. Let

$$\begin{aligned} D_{l,\delta} = \{\omega\colon \quad & x(t, \omega) = x,\ y(\tau, \omega) = i,\ \tau \in [t, \tau_{l+1} - \delta]; \\ & y(\tau, \omega) = j,\ \tau \in [\tau_{l+1}, t + \alpha]; \text{ one jump on } (\tau_{l+1} - \delta, \tau_{l+1}]\} \end{aligned}$$

Let $D_l = C_l - D_{l,\delta}$. By the Markov property, the infinitesimal conditions, and Theorem 1, for each $\epsilon > 0$ there exists a $\delta > 0$ for which

$$P(D_{l,\delta} | t, x, i) < \epsilon$$

Choosing $\epsilon$ as such, let $\{t_k{}^n\}$ be the $n$th binary partition of $[\tau_l, \tau_{l+1} - \delta]$ where $n$ is chosen so that $(\tau_{l+1} - \delta - \tau_l)/2^n < \delta$. Define

$$\begin{aligned} A_k{}^n = \Bigg\{\omega\colon \quad & x(\tau_l, \omega) = x + \int_t^{\tau_l} f(x_s, i)\, ds,\ y(\tau, \omega) = i,\ \tau \in [\tau_l, t_k{}^n]; \\ & y(\tau, \omega) = j,\ \tau \in [t_{k+1}^n, t + \alpha]; \text{ one jump on } (t_k{}^n, t_{k+1}^n]\Bigg\} \end{aligned}$$

Note that the $A_k{}^n$ are disjoint and $D_l = \bigcup_k A_k{}^n$. Let

$$\begin{aligned} B = \Bigg\{\omega\colon \quad & x(\tau_l, \omega) = x + \int_t^{\tau_l} f(x_s, i)\, ds,\ y(\tau, \omega) = i,\ \tau \in [\tau_l, t_k{}^n]; \\ & \text{one jump to } j \text{ on } (t_k{}^n, t_{k+1}^n]\Bigg\} \end{aligned}$$

Let $I_B$ be the indicator junction of the set $B$.

Suppose that $j$ is a rest phase. By the strong Markov property and Theorem 1

$$P(A_k{}^n | \tau_l, x_{\tau_l}, i) = E(I_B \exp\{-[t + \alpha - \gamma]\lambda_{jj}(x_\gamma)\} | \tau_l, x_{\tau_l}, i) \qquad \text{(B.1)}$$

where $\gamma$ is the random jump time on $(t_k{}^n, t_{k+1}^n]$. By the continuity of $\lambda_{jj}$ for $j$ a rest phase

$$\begin{aligned} E(I_B \exp\{-[t+\alpha-\gamma]\lambda_{jj}(x_\gamma)\}|\tau_l, x_{\tau_l}, i) & \\ \leqslant \exp\left\{-[t+\alpha-t_{k+1}^n]\min_{\rho\epsilon\phi_k}\lambda_{jj}(\rho)\right\}E(I_B|\tau_l, x_{\tau_l}, i) & \qquad \text{(B.2)} \\ \geqslant \exp\left\{-[t+\alpha-t_k{}^n]\max_{\rho\epsilon\phi_k}\lambda_{jj}(\rho)\right\}E(I_B|\tau_l, x_{\tau_l}, i) & \end{aligned}$$

where $\phi_k = [x_{t_k^n}, x_{t_{k+1}^n}]$.

By the Markov property and Theorem 1

$$E(I_B|\tau_l, x_{\tau_l}, i) = \exp\left[-\int_{\tau_l}^{t_n^k}\lambda_{ii}(x_s)\,ds\right]\left[\lambda_{ij}(x_{t_k^n})\,\Delta t + o(\Delta t)\right] \qquad \text{(B.3)}$$

where $\Delta t = (\tau_{l+1} - \delta - \tau_l)/2^n$.

Again by the Markov property and Theorem 1

$$P(D_l|t, x, i) = \sum_{k=0}^{2^n-1}\exp\left[-\int_t^{\tau_l}\lambda_{ii}(x_s)\,ds\right]P(A_k{}^n|\tau_l, x_{\tau_l}, i) \qquad \text{(B.4)}$$

By (B.1)–(B.3), passage to the limit as $n \to \infty$, and the arbitrariness of $\epsilon$, we conclude from continuity of $\lambda_{jj}$ and that of $\lambda_{ij}$ on $(x_{\tau_l}, x_{\tau_{l+1}}]$ that

$$P(C_l|t, x, i) = \int_{\tau_l}^{\tau_{l+1}}\exp\left[-\int_t^s\lambda_{ii}(x_s)\,ds\right]\lambda_{ij}(x_s)\exp[-(t+\alpha-s)\lambda_{jj}(x_s)]\,ds$$

Since $P(C|t, x, i) = \sum_l P(C_l|t, x, i)$, the result follows for $j$ a rest phase.

Suppose that $j$ is a forward phase. By the strong Markov property and the corollary to Theorem 1

$$P(A_k{}^n|\tau_l, x_{\tau_l}, i) = E\left\{I_B\exp\left[-\int_{x_\gamma}^{x_{t+\alpha}^*}\frac{\lambda_{jj}(v)}{f(v,j)}\,dv\right]\middle|\tau_l, x_{\tau_l}, i\right\}$$

where $\gamma$ is the random jump time on $(t_k{}^n, t_{k+1}^n]$ and $x_{t+\alpha}^*$ is defined in the theorem statement.

$$\begin{aligned} P(A_k{}^n|\tau_l, x_{\tau_l}, i) &\leqslant \exp\left[-\int_{x_{t_{k+1}^n}}^{\bar{x}_{t+\alpha-f_{\max}\Delta t}}\frac{\lambda_{jj}(v)}{f(v,j)}\,dv\right]E(I_B|\tau_l, x_{\tau_l}, i) \\ &\geqslant \exp\left[-\int_{x_{t_k^n}}^{\bar{x}_{t+\alpha+f_{\max}\Delta t}}\frac{\lambda_{jj}(v)}{f(v,j)}\,dv\right]E(I_B|\tau_l, x_{\tau_l}, i) \end{aligned}$$

where $\bar{x}_r$ is the solution to

$$\bar{x}_r = x_{t_{k+1}^n} + \int_{t_{k+1}^n}^r f(\bar{x}_s, j)\,ds \qquad \text{for} \quad r \geqslant t_{k+1}^n$$

and $\Delta t = (\tau_{l+1} - \delta - \tau_l)/2^n$. Note that for every $\omega \in A_k{}^n$,

$$\bar{x}_{t+\alpha} - f_{\max}\,\Delta t \leqslant x(t+\alpha, \omega) \leqslant \bar{x}_{t+\alpha} + f_{\max}\,\Delta t$$

Changing variables, we obtain

$$P(A_k{}^n|\tau_l, x_{\tau_l}, i) \leqslant \left(\exp \frac{\lambda_{\max} f_{\max}\,\Delta t}{f_{\min}}\right) \exp\left[-\int_{t^n_{k+1}}^{t+\alpha} \lambda_{jj}(\bar{x}_r)\,dr\right] E(I_B|\tau_l, x_{\tau_l}, i)$$

$$\geqslant \left(\exp \frac{-2\lambda_{\max} f_{\max}\,\Delta t}{f_{\min}}\right) \exp\left[-\int_{t^n_{k+1}}^{t+\alpha} \lambda_{jj}(\bar{x}_r)\,dr\right] E(I_B|\tau_l, x_{\tau_l}, i) \quad \text{(B.5)}$$

For the present subcase (B.3) and (B.4) are in effect. Hence by (B.3)–(B.5), passage to the limit as $n \to \infty$, the arbitrariness of $\epsilon$, and the continuity of $\lambda_{ij}(x_s)$ on $[\tau_l, \tau_{l+1})$, we conclude that

$$P(C_l|t, x, i) = \int_{\tau_l}^{\tau_{l+1}} \exp\left[-\int_t^s \lambda_{ii}(x_s)\,ds\right] \lambda_{ij}(x_s) \exp\left[-\int_s^{t+\alpha} \lambda_{jj}(x_\beta{}^*)\,d\beta\right] ds$$

Since $P(C|t, x, i) = \sum_l P(C_l|t, x, i)$, the result follows.

For $j$ a backward phase the proof is similar to the case of $j$ forward with the appropriate changes.

We remark that the case of $i$ a rest phase is a subcase the case of $i$ a forward phase.

For $i$ a backward phase the proof is similar to the above with the appropriate modifications.

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